Digital LTI system
Analog LTI system
Sampling
Decimation
Seismometer → Amplifier → AAA filter → DAA filter
Analog LTI system
Digital LTI system
Filtering (Digital Systems)

Convolution of Sequences
\[ y[n] = h[n] * x[n] \]

Discrete Fourier Transform (DFT)
\[ \tilde{Y}[k] = \tilde{T}[k] \tilde{X}[k] \]

\[ \tilde{X}[k] \quad \tilde{T}[k] \quad \tilde{Y}[k] = \tilde{T}[k] \tilde{X}[k] \]

z-Transform
\[ Y[z] = T[z]X[z] \]

\[ X[z] \quad T[z] \quad Y[z] = T[z]X[z] \]
The z-Transform

\[ Z\{x[n]\} = \sum_{n = -\infty}^{\infty} x[n]z^{-n} = X(z) \]

Properties

- \( x_1[n] * x_2[n] = \sum_{m = -\infty}^{\infty} x_1[m]x_2[n - m] \Leftrightarrow X_1(z) \cdot X_2(z) \) (convolution theorem)
- \( x[n - n_0] \Leftrightarrow z^{-n_0}X(z) \) (shifting theorem)
- \( x[-n] \Leftrightarrow X(1/z) \)

Transfer Function

\[ T(z) = \frac{Z\{y[n]\}}{Z\{x[n]\}} = \frac{Y(z)}{X(z)} \]
Rational Transfer Function $\Leftrightarrow$ Linear Difference Equation

$$T(z) = \frac{Y(z)}{X(z)} = \sum_{l=0}^{M} b_l z^{-l} \quad \Leftrightarrow \sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$
Recursive/Non-Recursive Filters

\[
\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]
\]

\[
a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]
\]

\[
y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} \frac{b_l}{a_0} x[n-l]
\]

- recursive
- non-recursive

( IIR Filters )

FIR Filters

\[
\begin{align*}
& a_0 = 1 \quad \text{and} \quad a_k = 0 \quad \text{for} \quad k \geq 1 \\
& y[n] = \sum_{l=0}^{M} b_l x[n-l]
\end{align*}
\]
Rational Transfer Function

\[ T(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}} \]

Special Case: FIR filter \((a_0 = 1\) and \(a_k = 0\) for \(k \geq 1\))

\[ T(z) = \sum_{l=0}^{M} b_l z^{-l} = b_0 \prod_{l=1}^{M} (1 - c_l z^{-1}) \]

\[ = z^{-M} \cdot b_0 \prod_{l=1}^{M} (z - c_l) \]

Linear Difference Equation

\[ y[n] = \sum_{l=0}^{M} b_l x[n - l] = x[n] \ast b[l] \]
• **FIR filters**:
  + Always stable.
  - Steep filters need many coefficients.
  + Both causal and noncausal filters can be implemented.
  + Filters with given specifications are easy to implement!

• **IIR filters**:
  - Potentially unstable and subject to quantization errors.
  + Steep filters can easily be implemented with a few coefficients. Speed.
  - Filters with given specifications are in general, difficult, if not impossible, to implement *exactly(!)*.
Recursive/Non-Recursive Filters

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l] \]

\[ a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l] \]

\[ y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{l=0}^{M} \frac{b_l}{a_0} x[n-l] \]

How can we remove these effects?

Back to Transfer function:

Recursive

non-recursive

( IIR Filters )

FIR Filters

\( a_0 = 1 \) and \( a_k = 0 \) for \( k \geq 1 \)

\[ y[n] = \sum_{l=0}^{M} b_l x[n-l] \]
Waveform Properties and Root Positions

- Minimum delay/phase
- Maximum delay/phase
- Mixed delay/phase (zero phase)
- Mixed delay/phase
- Mixed delay/phase
- Mixed delay/phase

Flip time axis
Zero Phase FIR Filter

**Problem:** Two-Sided IR

**Cure:** Change IR into Minimum Phase

**Methods:**

1) Add phase of Minimum Phase Filter to trace spectrum

2) Recursive Filtering of time inverted trace
Removing the acausal response of a Zero Phase FIR Filter

Linear Phase and Zero Phase Filter:

Linear-phase FIR Filter: \(F(z)\)

Zero-phase FIR Filter: \(F(z) = F(z) \cdot z^{lp}\)

\(z^{lp} = \text{Time delay correction by } lp \text{ samples}\)
General rule:
Any filter can be expressed by convolution of its minimum phase and maximum phase component.

roots within UC  roots outside UC

\[ F(z) = F_{\text{max}}(z) \cdot F_{\text{min}}(z) \]

maximum phase component -> left-sided (acausal) component
minimum phase component -> right-sided (causal) component
Removal of acausal response:

Replace maximum phase component $F_{\text{max}}(z)$ by its minimum phase equivalent $\text{MinPhase}\{F_{\text{max}}(z)\}$.

\[ \text{MinPhase}\{F_{\text{max}}(z)\} = \]

Answer: flip maximum phase component in time.
In terms of $z$-transform:

\[ \Rightarrow \text{MinPhase}\{F_{\text{max}}(z)\} = F_{\text{max}}(1/z) \]
z-transform representation of digital seismogram

\[ \tilde{Y}(z) = F(z) \cdot z^{lp} \cdot \tilde{X}(z) \]

digital seismogram after FIR filtering but before decimation

\[ \tilde{x}[n] = \text{unfiltered digital seismogram} \]
\[ \tilde{X}(z) = \text{z-transform of the input signal } \tilde{x}[n] \]
\[ F(z) = \text{z-transform of the linear phase FIR filter} \]
\[ \tilde{y}[n] = \text{filtered digital seismic trace before decimation} \]
\[ \tilde{Y}(z) = \text{z-transform of } \tilde{y}[n] \]

\[ ^1 \text{This is a fictitious signal since decimation is commonly done while filtering.} \]
$F(z) \cdot z^{lp}$ corresponds to a zero phase filter in which the linear phase component of $F(z)$ is corrected.

In practice: Treat time shift $z^{lp}$ separately from $F(z)$.
Removing the maximum phase component of a FIR filter

Principle:

\[ F_{\text{max}}(z) \rightarrow \text{MinPhase}\{ F_{\text{max}}(z) \} \]

‘Corrected’ seismogram \( Y(z) \):

\[ Y(z) = \frac{1}{F_{\text{max}}(z)} \cdot F_{\text{max}}(1/z) \cdot \tilde{Y}(z) \]

Problem: Since \( F_{\text{max}}(z) \) has only zeros outside the unit circle, \( 1/F_{\text{max}}(z) \) will have poles outside the unit circle.
Solution: Flip time axis.

\[ Y(1/z) = \frac{1}{F_{\max}(1/z)} \cdot F_{\max}(z) \cdot \tilde{Y}(1/z) \]

Impulse response corresponding to \( 1/ F_{\max}(z) \) becomes a stable causal sequence in nominal time and the deconvolution of the maximum phase component \( F_{\max}(z) \) poses no stability problems.
The difference equation

\[ F_{max}(1/z) \cdot Y(1/z) = F_{max}(z) \cdot \tilde{Y}(1/z) \]

Rewrite to

\[ A'(z) \cdot Y'(z) = B'(z) \cdot X'(z) \]

\[ A'(z) \iff F_{max}(1/z) \quad Y'(z) \iff Y(1/z) \]

\[ B'(z) \iff F_{max}(z) \quad X'(z) \iff \tilde{Y}(1/z) \]

Written as convolution sum:

\[
\sum_{k=-\infty}^{\infty} a'[k] \cdot y'[i-k] = \sum_{l=-\infty}^{\infty} b'[l] \cdot x'[i-l]
\]
Assumption: $F(z)$ contains $mx$ zeros outside the unit circle => wavelets $a'[k]$ and $b'[k]$ will be of length $mx + 1$.

\[
\sum_{k=0}^{mx} a'[k] \cdot y'[i - k] = \sum_{l=0}^{mx} b'[l] \cdot x'[i - l]
\]

Rearrange to

\[
y'[i] \cdot a'[0] + \sum_{k=1}^{mx} a'[k] \cdot y'[i - k] = \sum_{l=0}^{mx} b'[l] \cdot x'[i - l]
\]

which is equivalent to

\[
y'[i] = -\sum_{k=1}^{mx} \frac{a'[k]}{a'[0]} \cdot y'[i - k] + \sum_{l=0}^{mx} \frac{b'[l]}{a'[0]} \cdot x'[i - l]
\]
This is
\[ a[k] = \frac{a'[k]}{a'[0]} = \frac{f_{max}[mx-k]}{f_{max}[mx]} \quad \text{for } k = 1 \text{ to } mx \]

and
\[ b[l] = \frac{b'[l]}{a'[0]} = \frac{f_{max}[l]}{f_{max}[mx]} \quad \text{for } l = 0 \text{ to } mx \]

The convolution sum
\[ y'[i] = \sum_{k=1}^{mx} a[k] \cdot y'[i-k] + \sum_{l=0}^{mx} b[l] \cdot x'[i-l] \]

\[ y'[i] = \text{the time reversed ‘corrected’ sequence.} \]

To obtain \( y[i] \) flip \( y'[i] \) back in time.
‘Corrected’ seismogram: \[ Y(z) = \tilde{F}(z) \cdot \tilde{Y}(z) \]

\[ \tilde{F}(z) = \text{changes the linear phase filter } F(z) \text{ into a minimum phase filter} \]

\[ \Rightarrow \text{onset of output signal is advanced by } lp \text{ samples.} \]
To account for this time shift in the corrected seismogram:

a) change time tag of ‘corrected’ trace

or

b) delay ‘corrected’ trace by \( lp \) samples.
What is needed for ‘correction’?

\(mx + 1\) coefficients of the maximum phase portion of the linear phase FIR filter

How to get maximum phase component?

\[F(z) = \sum_{l=0}^{m} b_l z^{-l} = b_0 \prod_{l=1}^{m} (1 - c_l z^{-l})\]

\(mn\) zeros inside UC \((c_i^{\text{min}}) \leftrightarrow F_{\text{min}}(z)\)

\(mx\) zeros outside UC \((c_i^{\text{max}}) \leftrightarrow F_{\text{max}}(z)\)

\[F_{\text{max}}(z) = \sum_{i=0}^{mx} f_i^{\text{max}} z^{-i} = b_0 \prod_{i=1}^{mx} (1 - c_i^{\text{max}} z^{-i})\]

From zeros outside UC -> polynomial \(F_{\text{max}}(z)\). 
\(f_{\text{max}}[l]\) for \(l = 0\) to \(mx\): coefficients of polynomial \(F_{\text{max}}(z)\).
Correction Procedure

Decimation stages

Full Correction (not always necessary):

interpolation $\rightarrow f_{n-1} \rightarrow$ correction for FIR filter stage $n$
interpolation $\rightarrow f_{n-2} \rightarrow$ correction for FIR filter stage $n-1$
  
  ... 

interpolation $\rightarrow f_1 \rightarrow$ correction for FIR filter stage 2
interpolation $\rightarrow F_s \rightarrow$ correction for FIR filter stage 1

finally: decimation back to $f_n$
SIL FIR PLUS LINEAR PHASE CORRECTION (100 Hz)
Conclusions

FIR filter generated precursory artefacts:

- can become impossible to be identified visually
- can have similar scaling properties as nucleation phases

Zero - phase FIR filters in general

- affect the determination of all onset properties (onset times, onset polarities)

Consequence

For the interpretation of onset properties (onset times, onset polarities, nucleation phases, etc.) the acausal response of the zero-phase FIR filter has to be removed

but not

for waveform analysis.