Digitizers and dynamic range

Reinoud Sleeman
ORFEUS Data Center
sleeman @ knmi.nl

IRIS / ORFEUS Workshop
Understanding and Managing
Information from Seismological Networks

28 Feb – 4 Mar 2005, Palmanova, Italy
Seismograph system

- ground velocity [m/s]
- Volt [V]
- Counts
- Background noise
  - Sensor response
  - Sensor noise
  - Clip level
- Delta Sigma Converter
- Digitizer noise
  - Clip level
  - Dynamic range
- Bit averaging by anti-alias FIR filters
- High sample rate (32 kHz) bit stream
- Filter characteristics
- Decimation (downsampling)
- Low sample rate (e.g. 40 Hz) data stream
Oversampled Delta-Sigma A/D Digitizer
(one-bit noise shaping converter)

Comperator: ADC or quantizer
Feedback: average of y follows the average of x
Integrator: accumulates the quantization error e over time
Pulse train: pulse density representation of x
Oversampling: increase of resolution
\[
\begin{align*}
\dot{s}_i &= x_i - q(u_i) \\
\dot{u}_i &= s_{i-1} + u_{i-1} \\
q(u_i) &= u_i + e_i \\
q(u_i) &= x_{i-1} + (e_i - e_{i-1}) \\
2\text{-nd order:} & \quad q(u_i) = x_{i-1} + (e_i - 2e_{i-1} + e_{i-2})
\end{align*}
\]
Assumption: quantization noise is white noise
1-bit ADC: quantization error spectrum (oversampling factor 2)

No feedback

First order feedback
Oversampling factor: 2

Analog input

Digital “pulse train” output
Oversampling factor: 5

Analog input

Digital “pulse train” output
Very high sampling rate ADC with very poor resolution (1 bit)

- Feedback ➔ quantization error reduction

What is the relation with “dynamic range” and “resolution”?
Dynamic range / resolution

- **No gain ranging:**
  - High dynamic range,
  - High resolution

- **Gain ranging:**
  - High dynamic range,
  - Low resolution

- **Instrument noise level**

- **Clip level**

Dynamic range is frequency dependent
Dynamic range of a digitizer — some theory:

Quantization:
\[ e = q(x) - x \]

Variance of error:
\[ e_{rms}^2 = \int_{-\infty}^{\infty} [q(x) - x]^2 p(e) de = \frac{1}{\Delta} \cdot \int_{-\Delta/2}^{\Delta/2} e^2 de = \frac{\Delta^2}{12} \]

Number of quantization levels:
\[ \frac{2A}{\Delta} = 2^n \]

Dynamic range:
(Benett, 1948)
\[ SNR = 10 \cdot \log \left( \frac{s_{rms}^2}{e_{rms}^2} \right) \]
\[ s_{rms}^2 = \frac{A^2}{2} \]
\[ SNR = 10 \cdot \log \left( \frac{A^2 / 2}{\Delta^2 / 12} \right) = 1.76 + n \cdot 6.02 \]

\[ \downarrow \]
6 dB per bit
\[ \text{SNR}(f) = 10 \cdot \log \left( \frac{\text{PSD}_{\text{max}}(f)}{\text{PSD}_{\text{min}}(f)} \right) \]

\[ \text{PSD}_{\text{min}} = 10 \cdot \log \left( \frac{\Delta^2}{12 \cdot f_{\text{Nyquist}}} \right) \]

\[ \text{PSD}_{\text{max}} = 10 \cdot \log \left( \frac{A^2}{2 \cdot f_{\text{Nyquist}}} \right) \]

\[ \text{SNR} = 10 \cdot \log \left[ \frac{V_{pp}^2}{2} \cdot \frac{1}{f_{\text{Nyquist}}} \right] - 10 \cdot \log \left[ \left( \frac{V_{pp}}{2^n} \right)^2 \cdot \frac{1}{12 \cdot f_{\text{Nyquist}}} \right] \]

rel. to 1 V^2/Hz

\[ V_{pp} \]: peak-to-peak input in V

\[ n \]: number of bits

\[ f_{\text{Nyquist}} \]: Nyquist frequency
Very high sampling rate ADC with very poor resolution (1 bit)

- Feedback ➔ quantization error reduction
- Oversampling ➔ higher resolution
- Bitstream averaging + decimation ➔ low sample rate

Improved dynamic range and high resolution at a lower effective sampling rate
Demonstration
Java application of a
delta-sigma modulator
How to measure the dynamic range of a datalogger

- **shorten the input and the record self noise**
  - ratio of maximum peak amplitude and clip level
  - specify frequency band:
    \[
    \frac{\text{RMS noise}}{\text{RMS full scale sine}}
    \]
  - PSD graph, as function of the frequency
- **common input signal**
  - PSD graph (cross spectral analysis)
Dynamic range = \frac{(RMS \text{ noise})}{(RMS \text{ full scale sine})}

Shortened inputs:

Q4120: \[ V_{pp} = 40 \, V \implies V_{rms} = 14.1 \, V \]

\[ dt = 0.05 \, s \]

0.01 – 8 Hz: RMS noise 0.8 uV (measured)

\begin{align*}
143.8 \, dB & \quad \text{rel to } 1 \, V^2 / \text{Hz} \\
& \quad \text{at } 20 \, \text{sps}
\end{align*}
Dynamic range = \((\text{RMS noise}) / (\text{RMS full scale sine})\)

Shortened inputs:

Q4120: \(V_{pp} = 40 \text{ V} \Rightarrow V_{rms} = 14.1 \text{ V}\)

\(dt = 0.05 \text{ s}\)

0.01 – 8 Hz: RMS noise 0.8 uV (measured)

\[ SNR = 10 \cdot \log\left(\frac{A^2}{2} / \frac{\Delta^2}{12}\right) = 1.76 + n \cdot 6.02 \rightarrow 23.7 \text{ bits} \]

\[ \text{rel to } 1 \text{ V}^2 / \text{ Hz at 20 sps} \]

143.8 dB
Cross spectral analysis
Linear noise-model

Assumptions

(1) $\int x \cdot n_1 = \int x \cdot n_2 = \int x \cdot n_3 = 0$

(2) $n_1$, $n_2$ and $n_3$ are uncorrelated
Auto/Cross correlation

**Time domain**

\[ \begin{align*}
    y_1 &= x \otimes h_1 + n_1 \\
    y_2 &= x \otimes h_2 + n_2 \\
    y_3 &= x \otimes h_3 + n_3 
\end{align*} \]

**Frequency domain**

\[ \begin{align*}
    Y_1 &= X \cdot H_1 + N_1 \\
    Y_2 &= X \cdot H_2 + N_2 \\
    Y_3 &= X \cdot H_3 + N_3 
\end{align*} \]

\[ \begin{align*}
    \text{corr} (y_1, y_1) &\iff
    Y_1 \cdot Y_1^* = X \cdot X^* \cdot H_1 \cdot H_1^* + N_1 \cdot N_1^* = \\
    &= P_{xx} \cdot H_1 \cdot H_1^* + P_{n1n1} = \\
    &= P_{y1y1} \quad \text{(Auto Power Spectrum)} \\
\end{align*} \]

\[ \begin{align*}
    \text{corr} (y_1, y_2) &\iff
    Y_1 \cdot Y_2^* = X \cdot X^* \cdot H_1 \cdot H_2^* + N_1 \cdot N_2^* = \\
    &= P_{xx} \cdot H_1 \cdot H_2^* + P_{n1n2} = \\
    &= P_{y1y2} \quad \text{(Cross Power Spectrum)} \\
\end{align*} \]
\[
\begin{align*}
\mathbf{P}_{y_{1y1}} &= \mathbf{Y1} \cdot \mathbf{Y1}^* = \mathbf{X} \cdot \mathbf{X}^* \cdot \mathbf{H1} \cdot \mathbf{H1}^* + \mathbf{N1} \cdot \mathbf{N1}^* \\
\mathbf{P}_{y_{1y2}} &= \mathbf{Y1} \cdot \mathbf{Y2}^* = \mathbf{X} \cdot \mathbf{X}^* \cdot \mathbf{H1} \cdot \mathbf{H2}^* + \mathbf{N1} \cdot \mathbf{N2}^* \\
\mathbf{P}_{y_{1y3}} &= \mathbf{Y1} \cdot \mathbf{Y3}^* = \mathbf{X} \cdot \mathbf{X}^* \cdot \mathbf{H1} \cdot \mathbf{H3}^* + \mathbf{N1} \cdot \mathbf{N3}^* \\
\mathbf{P}_{y_{2y2}} &= \mathbf{Y2} \cdot \mathbf{Y2}^* = \mathbf{X} \cdot \mathbf{X}^* \cdot \mathbf{H2} \cdot \mathbf{H2}^* + \mathbf{N2} \cdot \mathbf{N2}^* \\
\mathbf{P}_{y_{2y3}} &= \mathbf{Y2} \cdot \mathbf{Y3}^* = \mathbf{X} \cdot \mathbf{X}^* \cdot \mathbf{H2} \cdot \mathbf{H3}^* + \mathbf{N2} \cdot \mathbf{N3}^* \\
\mathbf{P}_{y_{3y3}} &= \mathbf{Y3} \cdot \mathbf{Y3}^* = \mathbf{X} \cdot \mathbf{X}^* \cdot \mathbf{H3} \cdot \mathbf{H3}^* + \mathbf{N3} \cdot \mathbf{N3}^* \\
\mathbf{P}_{y_{2y2}} / \mathbf{P}_{y_{1y2}} &= \mathbf{H2} / \mathbf{H1} + \mathbf{N2} \cdot \mathbf{N2}^* / \mathbf{P}_{y_{1y2}} = \mathbf{P}_{y_{2y3}} / \mathbf{P}_{y_{1y3}} + \mathbf{P}_{n2n2} / \mathbf{P}_{y_{1y2}} \\
\mathbf{P}_{n2n2} &= \mathbf{P}_{y_{2y2}} - [\mathbf{P}_{y_{1y2}} / \mathbf{P}_{y_{1y3}}] \cdot \mathbf{P}_{y_{2y3}} \\
\end{align*}
\]
\[ P_{n1n1} = P_{y1y1} - \left[ \frac{P_{y1y3}}{P_{y2y3}} \right] \cdot P_{y2y1} \]

\[ P_{n2n2} = P_{y2y2} - \left[ \frac{P_{y1y2}}{P_{y1y3}} \right] \cdot P_{y2y3} \]

\[ P_{n3n3} = P_{y3y3} - \left[ \frac{P_{y3y2}}{P_{y1y2}} \right] \cdot P_{y3y1} \]

Cross power spectrum of outputs 1 and 3

Power spectrum of 3\textsuperscript{rd} digitizer-noise
PSD estimation (Welsh, 1978):

- 50 % overlapping time sections (of 2048 samples at 20 Hz)
- tapering (Hanning window)
- auto/cross correlation
- Fourier transform
- averaging over the number of time sections
PSD of self-noise Q4120 measured with common STS-2 vertical signal (@ 20 sps)
PSD of self-noise Q4120 measured with common STS-2 vertical signal (@ 20 sps)

\[
PSD_{\text{model}}(f) = PSD_{23.6\text{bit}} + PSD_{24.7\text{bit}} \cdot \frac{1}{f^{1.55}}
\]
NLNM recorded by a noise-free STS-2

Quantenna Q4120 Noise model
From: Wielandt

\[ \text{PSD} = \text{RMS}^2 / \text{BW} \]

@1000 sec: \( \text{RMS} = 212 \text{ dB} \quad \text{PSD} = \text{RMS}^2 / \text{BW} \)

\( \text{PSD} = 180 \text{ dB} \)
Datalogger data processing

- Analog input $x$
- Integrator
- Comperator
- Quantization error $e$
- Pulse train $y$
- ...0010011101000111010....
- 32000 Hz
Minimum phase, causal

Linear phase, acausal

Zero phase, acausal

constant time shift
Removing acausal effects of the zero phase FIR filter

- FIR coefficients (polynomial coefficients) - Quanterra
- polynomial root finding (Jenkins/Traub, Muller, Newton) - Octave, Markus Lang, Mathematica
- replace maximum phase roots by minimum phase roots (c → 1/c*)
- get minimum phase equivalent FIR coefficients - Octave
- apply correction filter - Scherbaum
**IIR filter: relation between coefficients and poles/zeros**

\[ y_k = \sum_{m=0}^{M} a_m \cdot x_{k-m} \]

\[ Y(z) = \sum_{m=0}^{M} a_m \cdot z^{-m} \cdot X(z) \]

**Numerator coefficients**

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} a_m \cdot z^{-m}}{\sum_{k=0}^{K} b_k \cdot z^{-k}} = \frac{a_0 \cdot \prod_{m=1}^{M} (1 - c_m \cdot z^{-1})}{b_0 \cdot \prod_{k=1}^{K} (1 - d_k \cdot z^{-1})} \]

**Denominator coefficients**

\( a_m \): filter coefficients

**Time domain**

\( c_m \): Roots of polynomial (zeros)

\( d_m \): Roots of polynomial (poles)

**Z-domain**

complex variable: \( z = e^{s \cdot T} \)
Polynomial root finding
random noise addition