So far..

- Concepts for continuous LTI systems
- Computer exercises based on sequences of numbers (approximation was acceptable in context)

⇒ Intuitive grasp of concepts

- However: important differences between continuous and discrete systems.

From infinitely continuous to finite discrete

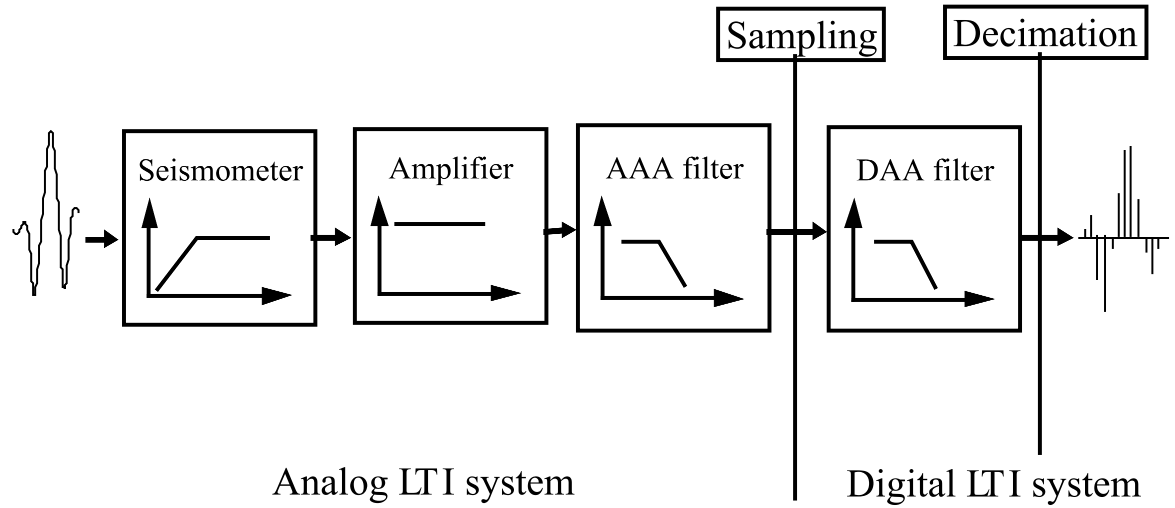


Fig. 7.1 Building blocks of modern seismic acquisition systems. AAA filter denotes the analog anti-alias filter, whereas DAA filter is the digital anti-alias filter to be applied before decimation.

- Modern DAS consist of analog and digital parts.

- Theoretical equivalent: analog and discrete systems

Sampling introduces a periodicity in the spectrum !

Can be seen from either

FT of product of impulse train and continuous signal

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\omega - kj\omega_s) \quad \text{with} \quad \omega_s = 2\pi / T$$

or

Definition of FT for discrete-time signals:

$$F\{x[nT]\} = X_d(j\omega) = \sum_{n=-\infty}^{\infty} x[nT]e^{-j\omega nT}$$

Replacing ω by $\omega + 2\pi/T$ $\Rightarrow$ $X_d\left(j\left(\omega + \frac{2\pi}{T}\right)\right) = X_d(j\omega)$

More general $X_d\left(j\left(\omega + \frac{2\pi}{T}r\right)\right) = X_d(j\omega)$ for any integer r.

Consequences

- In the discrete-time signal $x[nT]$ frequencies of ω and $\omega + (2\pi/T)$ can not be distinguished.

- For the synthesis we need to consider only a frequency range of length $2\pi/T$.

- **Inverse transform** (Kraniauskas, 1992):

$$F^{-1}\{X_d(j\omega)\} = x[nT] = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} X_d(j\omega)\, e^{j\omega nT} d\omega$$

Additional consequences

Since the frequency response function of a linear system equals the Fourier transform of the impulse response:

⇒ ***the frequency response function of a discrete-time system is always a periodic function with a period of*** $\omega_s = 2\pi / T$.

Different notation

(e. g. Oppenheim and Schafer [1989]):

$$F\{x[n]\} = X(e^{j\tilde{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\tilde{\omega}n}$$

and

$$F^{-1}\left\{X(e^{j\tilde{\omega}})\right\} = x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\tilde{\omega}}) e^{j\tilde{\omega}n} d\omega$$

Here, $x[n]$ replaces $x[nT]$ emphasizing the fact that $x[n]$ is merely a sequence of numbers which may or may not have been obtained by a physical sampling process. Furthermore, ωT is replaced by the digital frequency.

$$\tilde{\omega} = \frac{\omega}{F_{samp}} = \omega T$$

with the sampling frequency F_{samp}. In this notation the spectrum $X(e^{j\tilde{\omega}})$ becomes periodic with a period of 2π.

Notes

- Purpose of writing: $X(e^{j\tilde{\omega}})$ to indicate close relationship between the Fourier transform for discrete sequences and the z-transform.

  Here, ω will be used for analog frequency and $\tilde{\omega}$ for digital frequency.

- Asymmetry in Fourier pair corresponds to a modification of the equivalence between convolution and multiplication:

- Convolution of sequences corresponds to multiplication of periodic Fourier transforms, while multiplication of sequences corresponds to periodic convolution (with the convolution only carried out over one period) of the corresponding Fourier transforms.

7.6 The z-transform and the discrete transfer function

Analog components of DAS: **analog transfer function $T(s)$.**

Digital component: **discrete transfer function $T(z)$.**

While $T(s)$ was defined in terms of the Laplace transform, $T(z)$ is defined in terms of its discrete counterpart, **the z-transform**.

The bilateral z-transform

The **bilateral z-transform** of a discrete sequence $x[n]$ is defined as

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

The z-transform transforms the sequence $x[n]$ into a function $X(z)$ with z being a continuous complex variable.

The discrete transfer function

The **discrete transfer function**: z-transform of the output $Z\{y[n]\}$ divided by the z-transform of the input $Z\{x[n]\}$ of a discrete linear system.

$$T(z) = \frac{Z\{y[n]\}}{Z\{x[n]\}}$$

z-transform and Laplace transform for discrete signals

Apply Laplace transform to a sampled signal:

$$L\{x(t)\delta_T(t)\} = \int_{-\infty}^{\infty} x(t)\delta_T(t)e^{-st}\ dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\sum_{n=-\infty}^{\infty}\delta(t-nT)\right)e^{-st}\ dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)e^{-snT}$$

z-transform and Laplace transform for discrete signals, cont. 1

Result: $L\{x(t)\delta_T(t)\} = \sum_{n=-\infty}^{\infty} x(nT)e^{-snT}$

Formally replacing $x(nT)$ by $x[nT]$ yields Laplace transform for discrete-time sequences

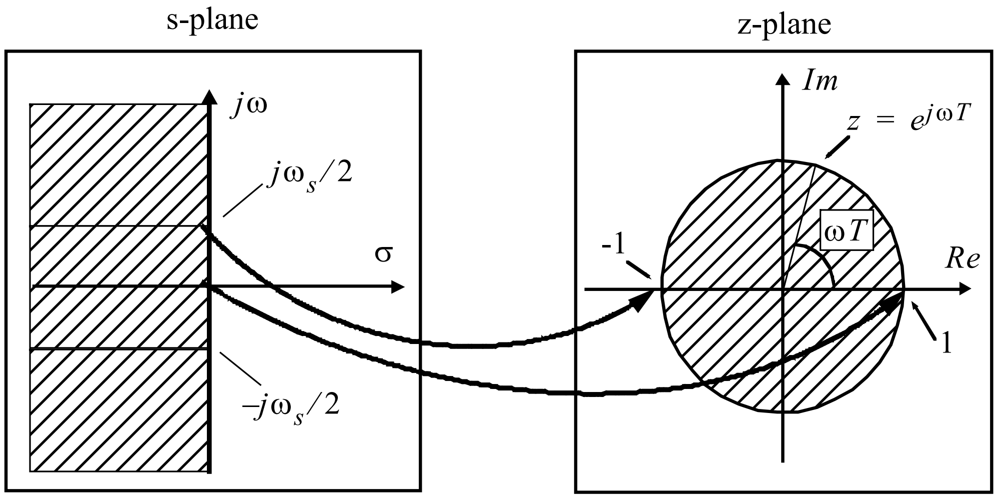
$$L\{x[nT]\} = \sum_{n=-\infty}^{\infty} x[nT]e^{-snT}$$

Here, $x[nT]$ denotes a discrete-time sequence for which the time interval between individual samples is T.

If we define $z = e^{sT}$ and $x[n] = x[nT]$ we obtain the definition of the z-transform:

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

Mapping of the complex s-plane onto the z-plane



Substitution rule

$$z = e^{sT} = e^{\sigma T} e^{j\omega T} = re^{j\omega T}$$

- The origin ($s = 0$) maps onto $z = 1$.
- $s = j\omega_s / 2 = j\,\pi/T$ maps onto $e^{j\pi} = \cos(\pi) = -1$.

- Since $z = e^{T\sigma} e^{j\omega T} = re^{j\omega T}$, the left half s-plane (the region with $s = \sigma + j\omega$ for $\sigma < 0$) maps to the interior of the unit circle in the z-plane ($r < 1$ because $e^{|\sigma| T} > 1$ and $r = e^{\sigma T} = e^{-|\sigma| T} = 1/e^{|\sigma| T}$).
- The right half s-plane is mapped onto the outside of the unit circle.

- For points on the imaginary axis ($s = j\omega$), r is 1 and z becomes the unit circle. Positive frequencies are mapped onto the upper half unit circle and negative frequencies onto the lower half. The complete linear frequency axis is wrapped around the unit circle with $\omega_s/2$ + multiples of 2π being mapped onto $z = -1$. For points on the unit circle ($z = e^{j\omega T}$), the z-transform reduces to the Fourier transform for discrete-time sequences $x[n]$.

FT and z-transform

The Fourier transform for discrete-time sequences equals the z-transform evaluated on the unit circle.

Continuous versus discrete

- **Continuous-time case**: Fourier transform equals the Laplace transform evaluated on the imaginary axis.

- **Discrete-time case:** Replace the role of the Laplace transform by the z-transform and the ‘role’ of the imaginary axis in the continuous-time case by the ‘role’ of the unit circle in the discrete-time case.

Intuitive deduction of z-transform properties

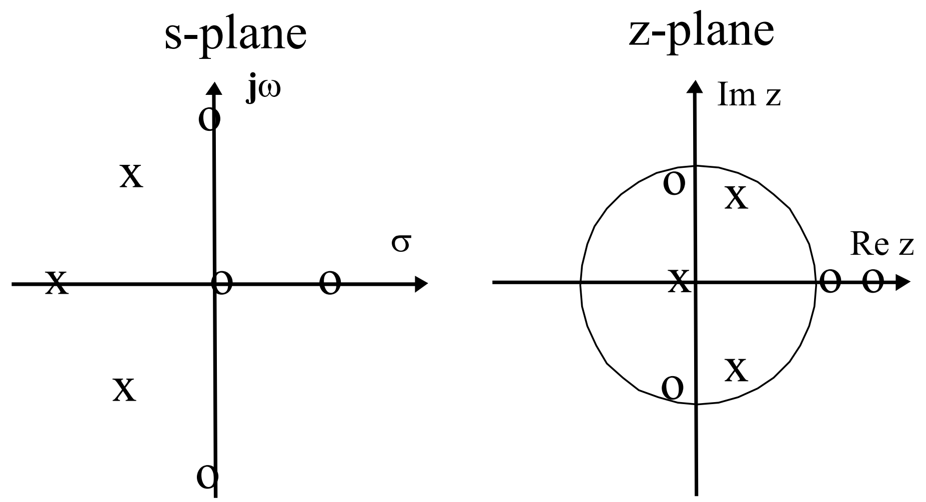


Fig. 7.5 Pole-zero distribution in s-plane and z-plane. The poles/ zeros in the s-plane close to will be mapped close to jω the unit circle. Poles/zeros on the real axis will stay on the real axis.

- Region of convergence of the z-transform and signal properties.

- Distribution of poles and zeros if the singularities (poles and zeros) are mapped according to $z = e^{sT}$. Singularities of transfer functions which lie in the left half s-plane are mapped to singularities inside the unit circle in z while singularities in the right halfplane map to the outside of the unit circle (Fig. 7.5).

z-transform properties

Shifting theorem $x[n-n_0] \Leftrightarrow z^{-n_0}X(z)$

Convolution theorem $x_1[n] * x_2[n] \Leftrightarrow X_1(z) \cdot X_2(z)$

In this context the convolution of two sequences is defined as

$$x_1[n]*x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

Replacing z by $1/z$ in the z-transform $X(z)$ corresponds to inverting the input signal ($x[n]$) in time

$$x[-n] \Leftrightarrow X(1/z)$$

General properties of discrete systems

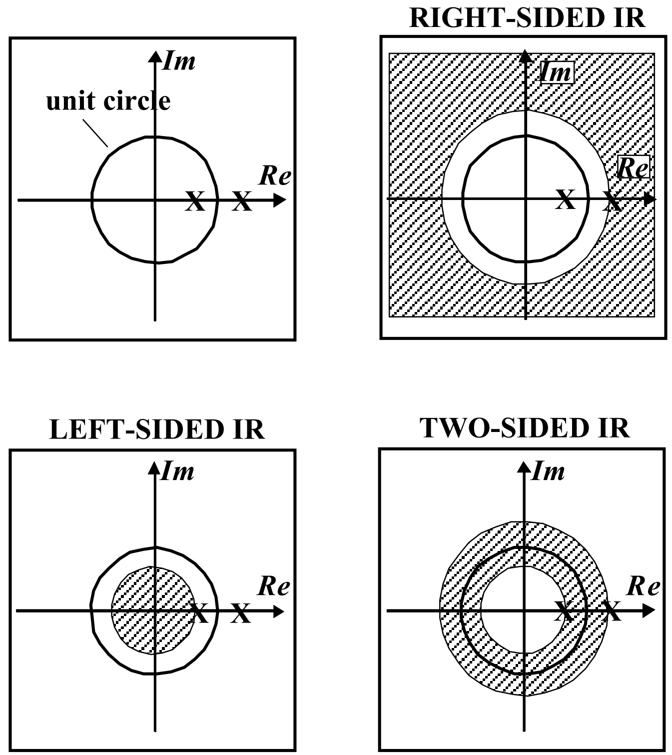


Fig. 7.5 Region of convergence and type of resulting impulse response functions (IR) for a discrete system containing two poles on the real axis.

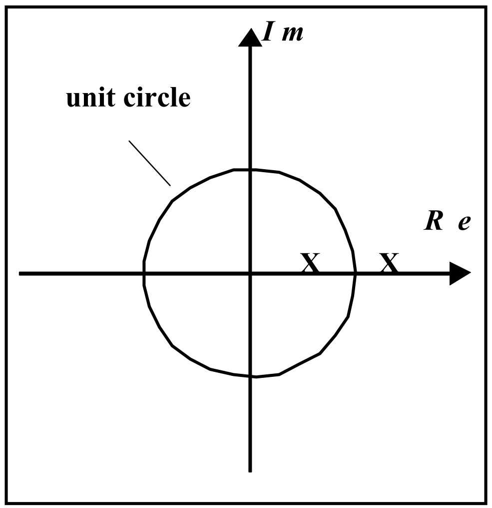
- The region of convergence (ROC) is a ring centered on the origin.
- The Fourier transform of $x[n]$ converges absolutely if and only if the unit circle is part of the ROC of the z-transform of $x[n]$.
- The ROC must not contain any poles.
- If $x[n]$ is a sequence of finite duration, the ROC is the entire z-plane except possibly $z = 0$ or $z = \text{inf}$.
- For a right-sided sequence the ROC extends outwards from the outermost pole to (and possibly including) $z = \text{inf}$.
- For a left-sided sequence the ROC extends inwards from the innermost pole to (possibly including) $z = 0$.
- For a two-sided sequence the ROC will consist of a ring bounded by poles on both the interior and the exterior.

Causality and Stability

For a discrete system to be **causal** and **stable**, all poles must lie inside the unit circle. If, in addition, the system contains no zeros outside the unit circle, it is called **minimum phase** system.

Problem 7.3

Which type of impulse response function would result if we want to use the inverse Fourier transform for evaluation of a system containing two poles on the real axis as shown in the Figure?



Problem 7.4

How do time shifts effect the region of convergence of the z-transform? Argue by using the shifting theorem for z-transforms ($x[n-n_0] <-> z^{-n_0}X(z)$) for positive and negative n_0.

7.7 The inverse z-transform

Formal evaluation by contour integral:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

with the closed contour C lying within a region of convergence and being evaluated counterclockwise.

If the ROC includes the unit circle and if the inverse z-transform is evaluated on it, the inverse z-transform reduces to the inverse Fourier transform.

<u>Other methods:</u>
Power series expansion or partial fraction expansion into simpler terms for which the inverse z-transforms are tabulated and can be looked up in a table (Proakis and Manolakis, 1992). Partial fraction expansion is particularly useful if the z-transform is a rational function in *z*!

Special case: Rational transfer function

Always the case if the system can be described by a **linear difference equations with constant coefficients:**

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

z-transform using the shifting theorem $x[n-n_0] \Leftrightarrow z^{-n_0} X(z)$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{l=0}^{M} b_l z^{-l} X(z)$$

Rational transfer function

for
$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{l=0}^{M} b_l z^{-l} X(z)$$

Discrete transfer function $T(z)$ becomes:

$$T(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{l=1} (1 - c_l z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$= z^{N-M} \cdot \left(\frac{b_0}{a_0}\right) \frac{\prod_{l=1}^{M} (z - c_l)}{\prod^{N} (z - d_k)}$$

which is a rational function in z.

c_l are the non-trivial zeros and d_k are the non-trivial poles of $T(z)$.

In addition $T(z)$ has $|N\text{-}M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin (trivial poles and zeros). Poles or zeros may also occur at $z = \inf$ if T(inf) = inf or 0, respectively.

Property rational transfer function

Note: Coefficients in transfer function are identical to the coefficients of the difference equation.

$$T(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{l=1} (1 - c_l z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$= z^{N-M} \cdot \left(\frac{b_0}{a_0}\right) \frac{\prod_{l=1}^{M} (z - c_l)}{\prod^{N} (z - d_k)}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

7.8 z-transform and Discrete Fourier Transform (DFT)

- CTFT: continuous-frequency domain
- DFT: discrete-frequency domain
- Transition from continuous-time to discrete-time corresponds to transition from aperiodic to periodic spectra,
- Transition from aperiodic signals to periodic signals: corresponds to_transition from continuous spectra to discrete spectra
- DTFT for infinite sequences equals the z-transform evaluated on the unit circle for continuous frequencies.
- DFT for $x[n]$ samples the z-transform of $\tilde{x}[n]$ at discrete frequencies, namely at N equally spaced points on the unit circle.

DFT representation in the z-plane

z-plane

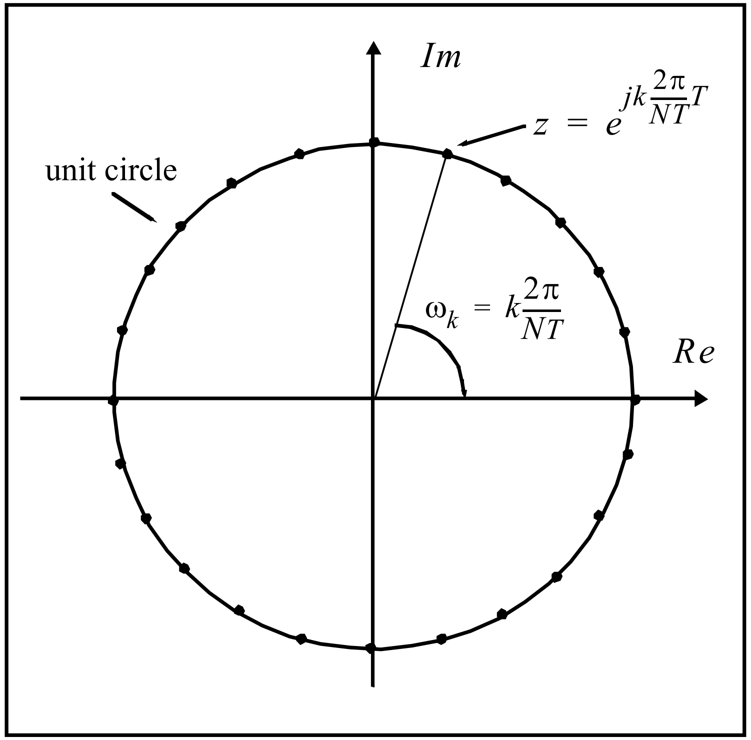


Fig. 7.6 DFT representation in the z-plane.

7.9 The convolution of sequences

Concept of the discrete transfer function *T*(*z*): z-transform of the output of a discrete filter can be calculated by multiplying the z-transform of the discrete filter input signal with $T(z)$.

Convolution theorem for the z-transform: this is equivalent to the discrete convolution of two sequences, namely the discrete filter input signal and the discrete impulse response of the filter.

By analogy we could therefore expect that this is true also for sequences of finite duration. In other words, that convolution could be done in the frequency domain by simply multiplying the corresponding discrete DFT spectra.

Is this correct?

Problem 7.5

Is it possible to perform convolution in the frequency domain by simply multiplying the corresponding discrete DFT spectra?
First, create the impulse response functions of two filters which perform just a simple time shift. The corresponding impulse response functions simply consist of time shifted impulses. For a sampling rate of 100 Hz and a time shift of 5.5 sec, the impulse response function would consist of a spike at sample 550. For a 6 sec delay filter, the spike would have to be located at sample 600. The convolution of the two filters should result in a time shifting filter which delays the input signal by 11.5 sec. Use 1024 points for the calculation of the FFT.

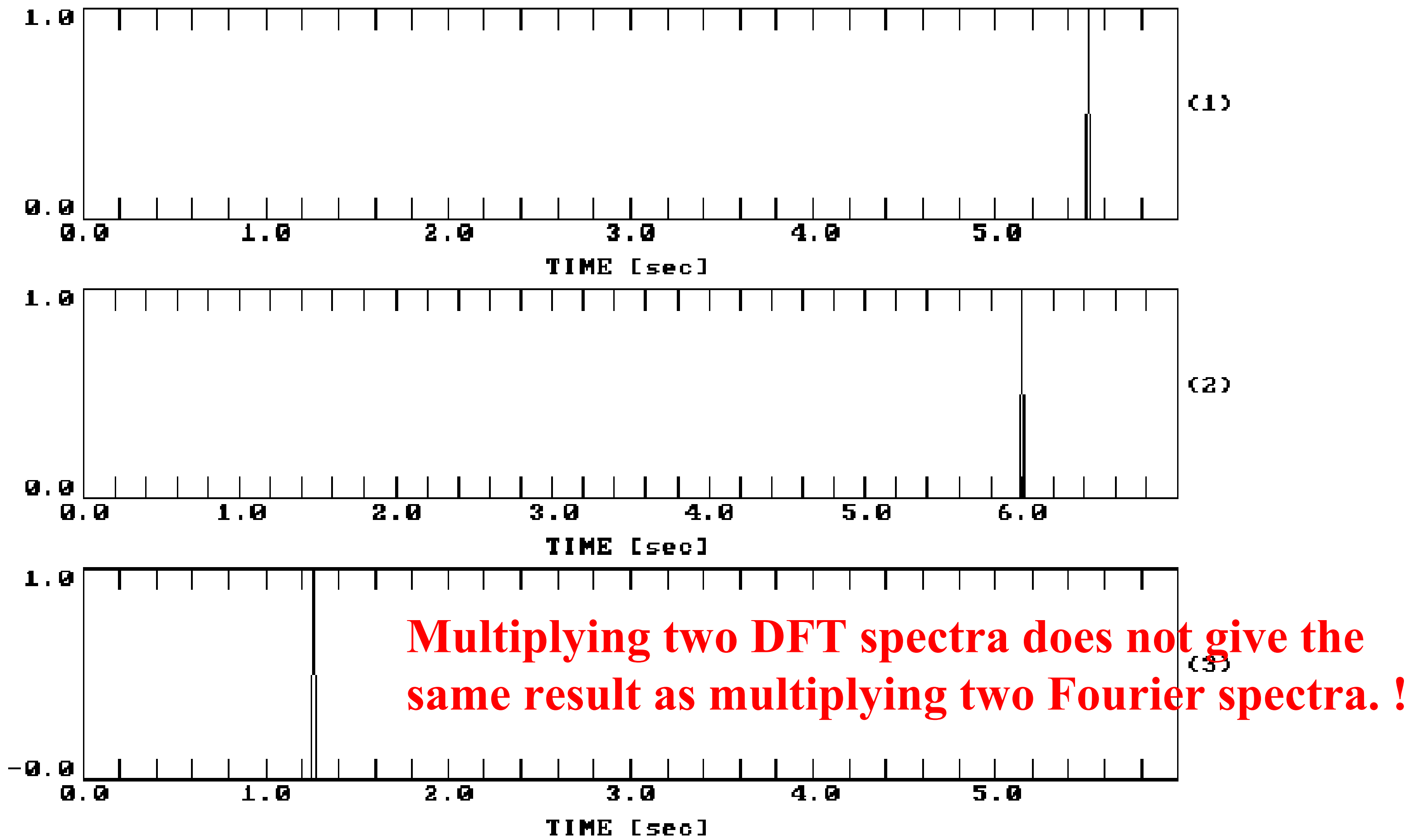


Fig. 7.7 The wrap around effect with discrete convolution in the frequency domain. Channel 3 is the inverse DFT of the product of the DFT spectra of channels 1 and 2. The no. of points used for calculating the DFT was equivalent to 10.24 sec

What happened?

***Wrap around effect* or *temporal aliasing*.** The DFT is essentially the Fourier series representation of an infinite periodic sequence $\tilde{x}[nT]$ which is the periodic extension of the given finite sequence $x[nT]$. Consequently, multiplying two DFT spectra corresponds to the convolution of two infinite periodic sequences.

Convolution theorem

Convolution theorem for periodic sequences (circular convolution)

If a sequence $\tilde{x}_1[n]$ is periodic with a period N then its discrete convolution with another signal $\tilde{x}_2[n]$ of duration N is identical to the inverse DFT of the product of the corresponding DFT spectra.

Intuition and the DFT....

The properties of the DFT can be intuitively derived from the corresponding properties of the Fourier transform, <u>**only**</u> **if we do not think in terms of the finite sequence** $x[n]$ but always in terms of $\tilde{x}[n]$ which contains infinitely many replicas of $x[n]$.

Trick to avoid the wrap around effect by *zero padding*: artificially increase the period (N) by padding the signal with trailing zeros to make it larger than the largest time lag for which the input signal could possibly be affected by the impulse response.

Problem 7.6

How many samples long should the zero padding be in the example above in order to eliminate the wrap around effect?